A coupling for prime factors of a random integer

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Joint work with Dimitris Koukoulopoulos

Large prime factors

- Pick an integer N uniformly in $[1, x]$.
- Let $P_1 P_2 \cdots$ be the unique factorization of N with $P_1 \geqslant P_2 \geqslant \cdots$ being all primes or ones.

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Theorem (Billinglsey, 1972)

For any fixed $r \geqslant 1$, the random vector

$$\left(\frac{\log P_1}{\log x}, \ldots, \frac{\log P_r}{\log x} \right) \xrightarrow{d} (V_1, \ldots, V_r)$$

where $\mathbf{V} := (V_1, V_2, \ldots)$ satisfy a **Poisson-Dirichlet distribution***.*

Poisson-Dirichlet distribution

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1. With the density of the first r components:

$$f(t_1, \ldots, t_r) = \frac{1}{t_1 \cdots t_r} \cdot \rho\left(\frac{1 - (t_1 + \cdots + t_r)}{t_r}\right)$$

for $t_1 \geqslant t_2 \geqslant \cdots \geqslant t_r \geqslant 0$ and $\sum_i t_i \leqslant 1$.

The function $\rho\colon \mathbb{R}_{\geqslant 0} \to \mathbb{R}$ is the **Dickman-de Bruijn function**, characterized by:

- ρ is continuous;
- $\rho(u) = 1$ if $u \in [0, 1]$;
- $u\rho'(u) = -\rho(u-1)$ for $u \geqslant 1$.

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- Sample $(U_i)_{i \geqslant 1}$ independently and uniformly in $[0, 1]$.
- Define
  - ⋆ $L_1 := U_1$;
  - ⋆ $L_i := U_i(1 - U_{i-1}) \cdots (1 - U_1)$ for $i \geqslant 2$.

  GEM distribution : Distribution of $\mathbf{L} = (L_1, L_2, \ldots)$.
- Let $\mathbf{V} = (V_1, V_2, \ldots)$ be the non-increasing order statistics of $\mathbf{L}$.
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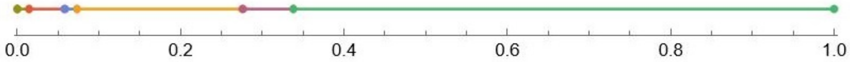

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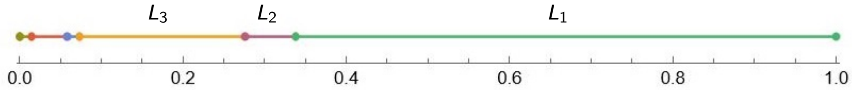

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Expected ℓ^1-distance

Theorem (Arratia, 2002)

$$\mathbb{E} \sum_{i \geqslant 1} |\log P_i - (\log x) V_i| \ll \log\log x.$$

Expected ℓ^1-distance

Definition

Let X and Y be two random variables on different probability spaces.

A **coupling** is a probability space over which there exists X' and Y' which have the same distribution as X and Y respectively.

Theorem (Arratia, 2002)

There exists a coupling of N and $\mathbf{V}$ satisfying

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He conjectured that there exists a coupling such that this expectation is $O(1)$.

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We cannot do better than $O(1)$ because $\mathbb{E}[\#\{i\geqslant 1 : V_i \in [a,2a]\}] = \log 2$ for all $a\in(0,\frac{1}{2})$.
Choose $a = \frac{\log 2}{3\log x}$, then $(\log x)V_i \in [a,2a]$ are all $\frac{\log 2}{3}$ away from any $\log p$.

1. Arratia's coupling refined

2. The Coupling Method: Random Factorizations

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Possible to choose Q_i's such that
$$\mathbb{P}[M \in A] = \frac{|A|}{\lfloor x \rfloor} + O\left(\frac{\log \log x}{\log x}\right) \text{ for any } A \subseteq \mathbb{N} \cap [1, x].$$

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Our modification:

- Choose $Q_i \in \{\text{prime powers}\} \cup \{1\}$ such that it is close to x^{L_i}.
- Set $J := \prod_{i \geqslant 2} Q_i$.
- Pick $P_{\texttt{extra}}$ randomly with $\mathbb{P}[P_{\texttt{extra}} = p | J = j] = \frac{\log p}{\theta(x/j)}$.
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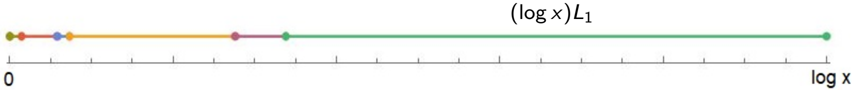
Possible to choose Q_i's such that
$$\mathbb{P}[M \in A] = \frac{|A|}{\lfloor x \rfloor} + O(\frac{1}{\log x}).$$

Constructing M



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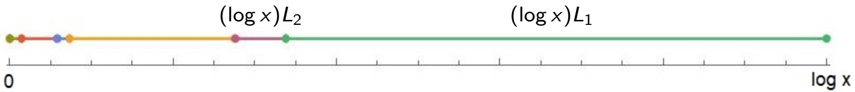

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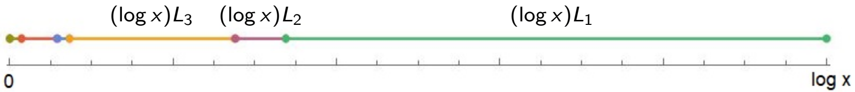

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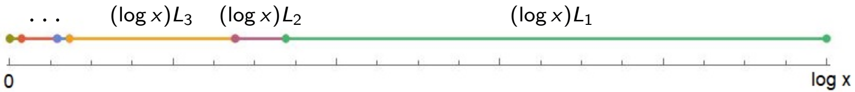

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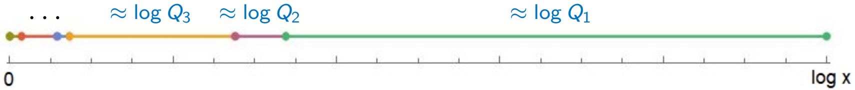

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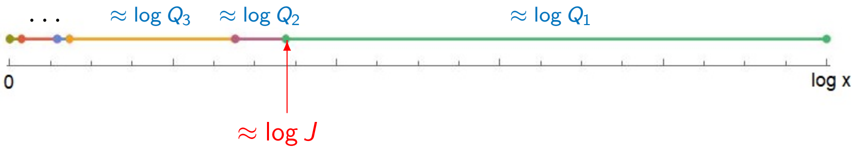

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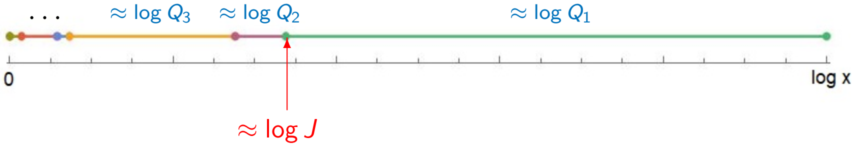

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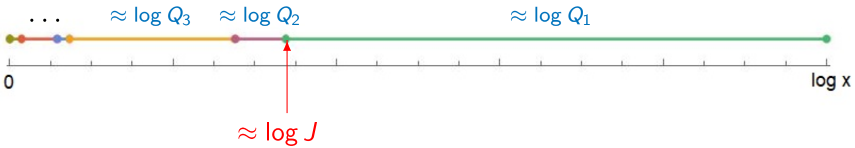

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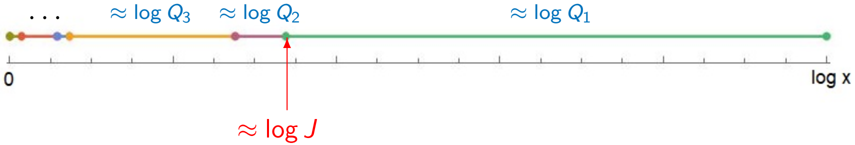

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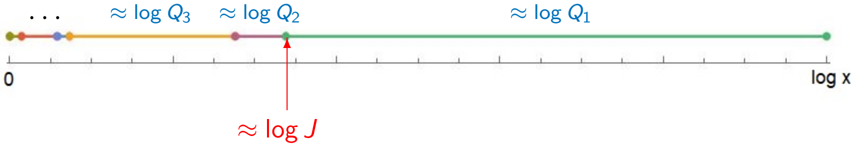

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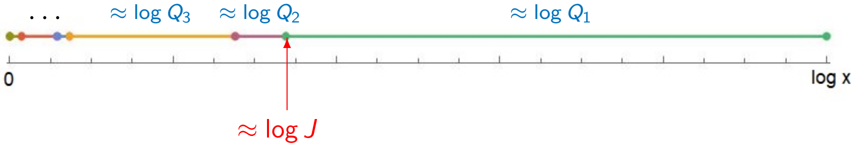

- For fixed d, we have $\mathbb{P}[d|J] \to \frac{1}{d}$ as $x \to \infty$. ✓
- But J is **far** from uniform ($\mathbb{P}[J \leqslant y] \approx \frac{\log y}{\log x}$) ✗
- So we need to insert a minimal number of extra primes to **adjust** the distribution. This is why we need an extra prime!

The total variation distance

The **total variation distance** between the laws of two real-valued random variables X and Y is

$$d_{\mathrm{TV}}(X, Y) := \sup_{A \subseteq \mathbb{R}} |\mathbb{P}[X \in A] - \mathbb{P}[Y \in A]|.$$

An equivalent definition is

$$d_{\mathrm{TV}}(X, Y) := \min \mathbb{P}[X \neq Y]$$

where the minimum is over all possible couplings of X and Y.

Recall, that we have $\mathbb{P}[M \in A] = \mathbb{P}[N \in A] + O(\frac{1}{\log x})$,

This implies we can construct a random integer M from N so that $\mathbb{P}[M \neq N] \ll \frac{1}{\log x}$.

Expected ℓ^1-distance

Let $\mathcal{E}$ be the event $M = N$ inside the coupling.

$$\mathbb{E} \sum_{i \geqslant 1} |\log P_i - (\log x) V_i| \leqslant \mathbb{E} \left[\mathbf{1}_{\mathcal{E}} \cdot \sum_{i \geqslant 1} |\log P_i - (\log x) V_i| \right] + (\log x) \cdot \mathbb{P}[\mathcal{E}^c]$$

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The P_i's are now the prime factors of M inside $\mathcal{E}$. We reduce this problem to bounding

$$\mathbb{E}\sum_{i\geqslant 2} |\log Q_i - (\log x)L_i| + \mathbb{E}[\log P_{\mathtt{extra}} - (\log x)L_1] + 2\cdot(\log x)\cdot\mathbb{P}[\mathcal{E}^c].$$

We can compute that this is $O(1)$.

1 Arratia's coupling refined

2 The Coupling Method: Random Factorizations

Predicting the size of divisors

Here are the **divisors** on a **logarithmic scale** of two distinct integers around 3.52155×10^{25} with exactly 32 divisors:

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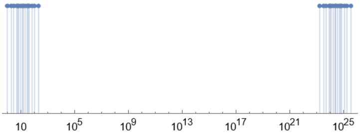

Figure: Divisors of $2 \times 3 \times 5 \times 7 \times 167693089728691213172671$

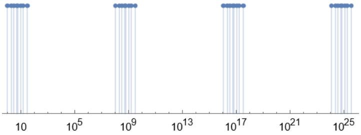

Figure: Divisors of $2 \times 3 \times 5 \times 105488303 \times 1112778700 8134317$

Average distribution of divisors

- Let N be a random integer uniform in $\mathbb{N} \cap [1, x]$.
- Let D be a random divisor of N uniformly distributed.

Theorem

For $x \geqslant 2$ and uniformly for $u \in [0, 1]$, we have

$$\mathbb{P}[D < N^u] = \frac{2}{\pi} \arcsin(\sqrt{u}) + \cdots$$

- *Deshouillers, Dress, Tenenbaum (1979):*
  $\cdots + O\left(\frac{1}{\sqrt{\log x}}\right)$ *for $u \in [0, 1]$.*
- *Bareikis, Manstavičius (2007):*
  $\cdots + O\left(\frac{1}{\sqrt{u(1-u)} \log x}\right)$ *for $u \in [\frac{1}{\log x}, 1 - \frac{1}{\log x}]$.*

Multiplicative Factorizations into k parts

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- *Bareikis, Manstavičius (2007):*

  Estimated $\mathbb{P}[D < N^u]$ when $\mathbb{P}[D = d \mid N = n] = f(d)$ for any $d|n$, and f being **multiplicative**, **fixed at primes**, satisfying $1 * f = 1$ and $f \geqslant 0$.

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- *Nyandwi, Smati (2013):*

  Estimated $\mathbb{P}[D_1 < N^{u_1}$ and $D_2 < N^{u_2}]$ when $D_1 D_2 D_3 = N$ is random 3-factorization (unif. dist.).

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- *Leung (2023):*

  Estimated $\mathbb{P}[D_1 < N^{u_1}, \ldots, D_{k-1} < N^{u_{k-1}}]$ for any k, for $\mathbb{P}[D_1 = d_1, \ldots, D_k = d_k \mid N = n] = f(d_1, \ldots, d_k)$ if $d_1 \cdots d_k = n$, and f is **multiplicative**, **fixed at primes**.

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Q: Can we use the distribution of large prime factors to get a similar result?

Probabilistic version:

Theorem (Donnelly and Tavaré, 1987)

- χ *random* k*-colouring of* $\mathbb{N}$*, with* $\mathbb{P}[\chi(i) = j] = \frac{1}{k}$.
- $Z_m := \sum_{i \geqslant 1} 1_{\chi(i)=j} V_i$ *for* $1 \leqslant m \leqslant k$.

Then $\mathbf{Z} := (Z_1, \ldots, Z_k) \sim \mathrm{Dir}(\frac{1}{k}, \ldots, \frac{1}{k})$.

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There exists a prob. space where $V_i \approx \frac{\log P_i}{\log x}$, we assign a random colour to each prime factor. This creates $\mathbf{D} := (\frac{\log D_1}{\log N}, \dots, \frac{\log D_k}{\log N})$. We have $\|\mathbf{D} - \mathbf{Z}\|_\infty \asymp \frac{1}{\log x}$ typically.

Theorem (Leung, 2023)

Let $\mathbf{u} \in [0,1]^{k-1}$ *with* $u_1 + \cdots + u_{k-1} \leqslant 1$.
Pick N *uniformly in* $[1,x]$. *Pick* $D_1 \cdots D_k = N$ *uniformly among all* k*-factorizations, then*

$$\mathbb{P}[D_1 < N^{u_1}, \ldots, D_{k-1} < N^{u_{k-1}}] = F(u_1, \ldots, u_{k-1}) + O_k\left(\frac{1}{(\log x)^{1/k}}\right).$$

Used variant of Landau-Selberg-Delange method

Factorizations into k parts

Theorem (H., Koukoulopoulos, 2024)

Let $\mathbf{u} \in (0,1)^{k-1}$*.*
Pick N *uniformly in* $[1, x]$*. Pick* $D_1 \cdots D_k = N$ *uniformly among all* k*-factorizations, then*

$$\mathbb{P}[D_1 < N^{u_1}, \ldots, D_{k-1} < N^{u_{k-1}}] = F(u_1, \ldots, u_{k-1}) + O\left(\sum_{1 \leqslant i < k} \frac{1}{u_i^{1-1/k}(1-u_i)^{1/k} \log x} \right).$$

Factorizations into k parts

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...and the fact that $(\frac{\log D_1}{\log N}, \ldots, \frac{\log D_k}{\log N})$ has discrete support (e.g.: $u_i < \frac{\log 2}{\log x}$ and $D_i < N^{u_i}$ implies $D_i = 1$).

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- Use this space as a bridge to pass from probabilistic model to get to information on stats of divisors!